Questions

Q1.

The function f is defined by

$$f(x) = 1 - \frac{2}{(x+4)} + \frac{x-8}{(x-2)(x+4)}, \quad x \in \mathbb{R}, x \neq -4, x \neq 2$$

(a) Show that $f(x) = \frac{x-3}{x-2}$

(5)

The function g is defined by

$$g(x) = \frac{e^x - 3}{e^x - 2}, \quad x \in \mathbb{R}, x \neq \ln 2$$

(b) Differentiate g(x) to show that g '(x) = $\frac{e^x}{(e^x - 2)^2}$

(3)

(c) Find the exact values of x for which g '(x) = 1

(4)

(Total 12 marks)

Q2.

$$f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \qquad x \neq \pm 3, x \neq -\frac{1}{2}$$

(a) Show that

$$f(x) = \frac{5}{(2x+1)(x+3)}$$

(5)

The curve C has equation y = f (x). The point $P\left(-1, -\frac{5}{2}\right)$ lies on C.

(b) Find an equation of the normal to C at P.

(8)

(Total 13 marks)

Q3.

$$f(x) = \frac{2x+2}{x^2-2x-3} - \frac{x+1}{x-3}$$

(a) Express f (x) as a single fraction in its simplest form.

(4)

(b) Hence show that f '(x) = $\frac{2}{(x-3)^2}$

(3)

(Total 7 marks)

Q4.

(a) Express

$$\frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)}$$

as a single fraction in its simplest form.

(4)

Given that

$$f(x) = \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2, \qquad x > 1,$$

(b) show that

$$f(x) = \frac{3}{2x-1}$$

(2)

(c) Hence differentiate f (x) and find f '(2)

(3)

(Total 9 marks)

Q5.

$$h(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \qquad x \geq 0$$

(a) Show that $h(x) = \dfrac{2x}{x^2+5}$

(4)

(b) Hence, or otherwise, find $h'(x)$ in its simplest form.

(3)

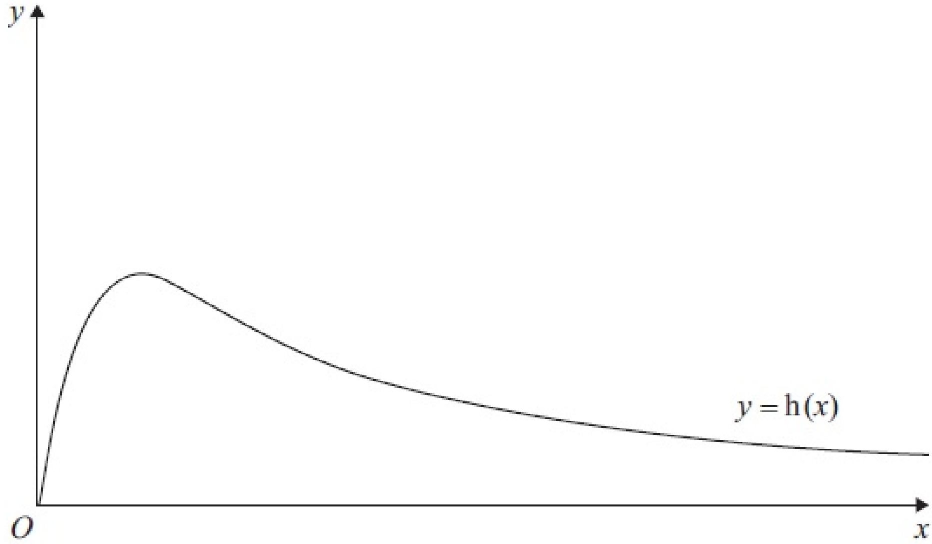


Figure 2

Figure 2 shows a graph of the curve with equation $y = h(x)$.

(c) Calculate the range of $h(x)$.

(5)

(Total 12 marks)

Q6.

The functions f and g are defined by

$$f : x \, {}^{x^2} \, 3x + \ln x, \qquad x > 0, \qquad x \in \mathbb{R}$$

$$g : x \, {}^{x^2} \, e\frac{d}{dx}, \qquad x \in \mathbb{R}$$

(a) Write down the range of g.

(1)

(b) Show that the composite function fg is defined by

$$\mathrm{fg} : x\ {}^{x^2} x2 + 3\frac{\mathrm{d}}{\mathrm{d}x}, \qquad x \in \mathbb{R}.$$

(2)

(c) Write down the range of fg.

(1)

(d) Solve the equation $\frac{\mathrm{d}}{\mathrm{d}x}$ [fg(x)] = x($xe\frac{\mathrm{d}}{\mathrm{d}x}$ + 2).

(6)

(Total 10 marks)

Q7.

A curve C has parametric equations

$$x = 2\sin t, \quad y = 1 - \cos 2t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $t = \frac{\pi}{6}$

(4)

(b) Find a cartesian equation for C in the form

$$y = \mathrm{f}(x), \ -k \leq x \leq k,$$

stating the value of the constant k.

(3)

(c) Write down the range of f(x).

(2)

(Total 9 marks)

Q8.

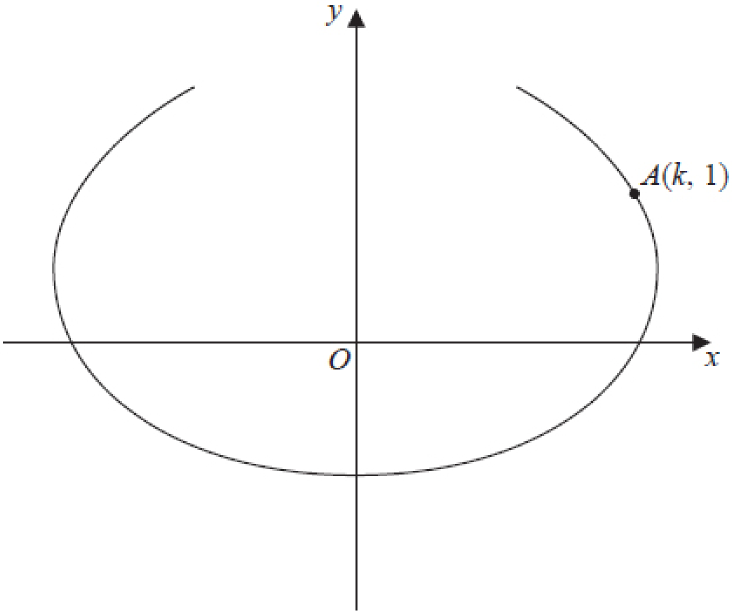


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = t - 4\sin t,\ y = 1 - 2\cos t, \quad -\tfrac{2\pi}{3} \leq t \leq \tfrac{2\pi}{3}$$

The point A, with coordinates $(k, 1)$, lies on the curve.

Given that $k > 0$

(a) find the exact value of k,

(2)

(b) find the gradient of the curve at the point A.

(4)

There is one point on the curve where the gradient is equal to $-\tfrac{1}{2}$

(c) Find the value of t at this point, showing each step in your working and giving your answer to 4 decimal places.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

(Total 12 marks)

Q9.

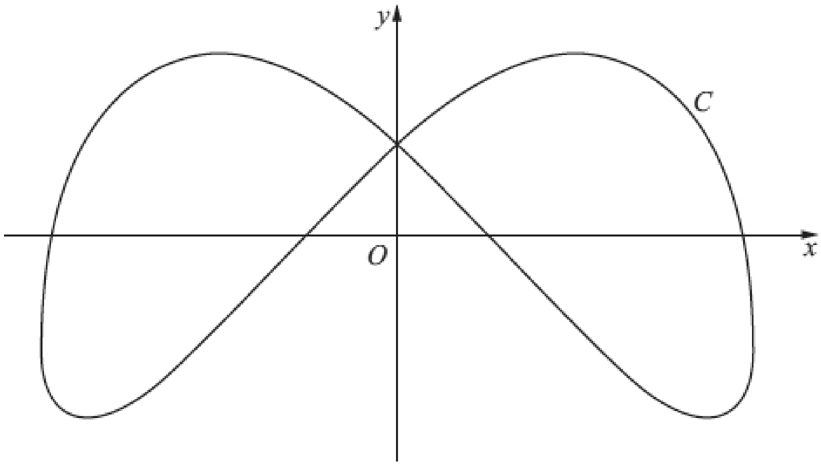


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4\sin\left(t + \frac{\pi}{6}\right), \quad y = 3\cos 2t, \quad 0 \leqslant t < 2\pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(3)

Find the coordinates of all the points on C where $\frac{dy}{dx} = 0$

(5)

(Total 8 marks)

Q10.

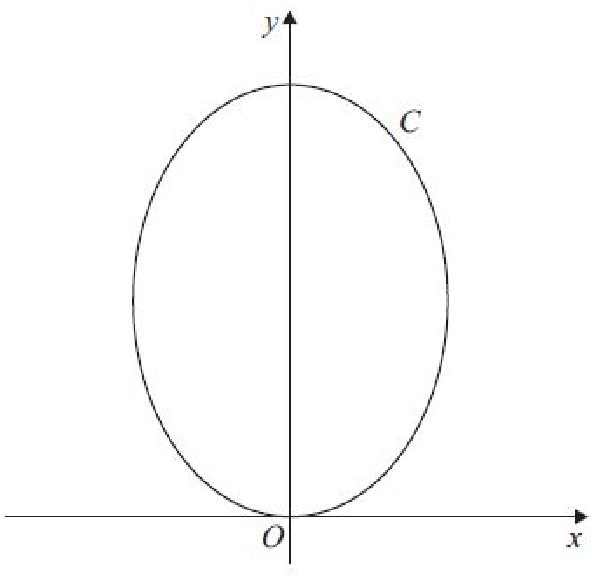


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = (\sqrt{3})\sin 2t, \quad y = 4\cos^2 t, \quad 0 \leq t \leq \pi$$

(a) Show that $\frac{dy}{dx} = k(\sqrt{3})\tan 2t$ where k is a constant to be determined.

(5)

(b) Find an equation of the tangent to C at the point where $t = \frac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants.

(4)

(c) Find a cartesian equation of C.

(3)

(Total 12 marks)

Q11.

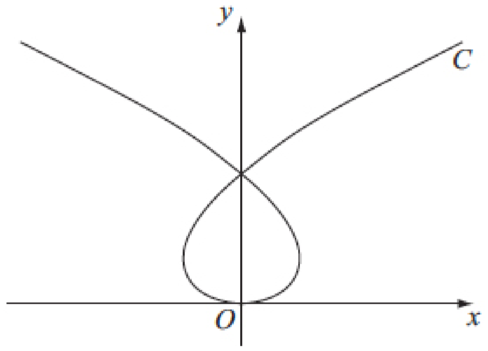


Figure 3

The curve C shown in Figure 3 has parametric equations

$$x = t^3 - 8t, \; y = t^2$$

where t is a parameter. Given that the point A has parameter $t = -1$,

(a) find the coordinates of A.

(1)

The line l is the tangent to C at A.

(b) Show that an equation for l is $2x - 5y - 9 = 0$.

(5)

The line l also intersects the curve at the point B.

(c) Find the coordinates of *B*.

(6)

(Total 12 marks)

Q12.

A curve *C* has parametric equations

$$x = \sin^2 t, \quad y = 2\tan t, \quad 0 \leqslant t < \frac{\pi}{2}$$

(a) Find $\frac{dy}{dx}$ in terms of t.

(4)

The tangent to *C* at the point where $t = \frac{\pi}{3}$ cuts the *x*-axis at the point *P*.

(b) Find the *x*-coordinate of *P*.

(6)

(Total 10 marks)

Q13.

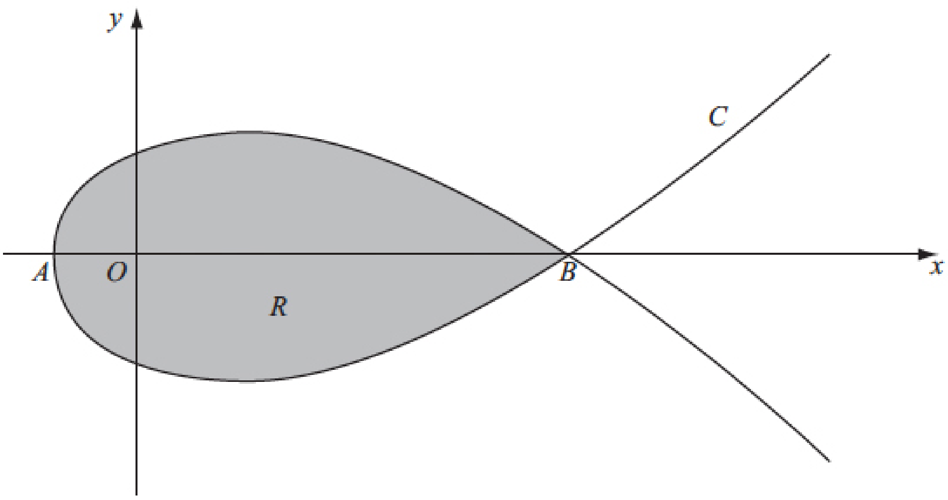


Figure 2

Figure 2 shows a sketch of the curve *C* with parametric equations

$$x = 5t^2 - 4, \qquad y = t(9 - t^2)$$

The curve *C* cuts the *x*-axis at the points *A* and *B*.

(a) Find the *x*-coordinate at the point *A* and the *x*-coordinate at the point *B*.

(3)

The region *R*, as shown shaded in Figure 2, is enclosed by the loop of the curve.

(b) Use integration to find the area of *R*.

(6)

(Total 9 marks)

Q14.

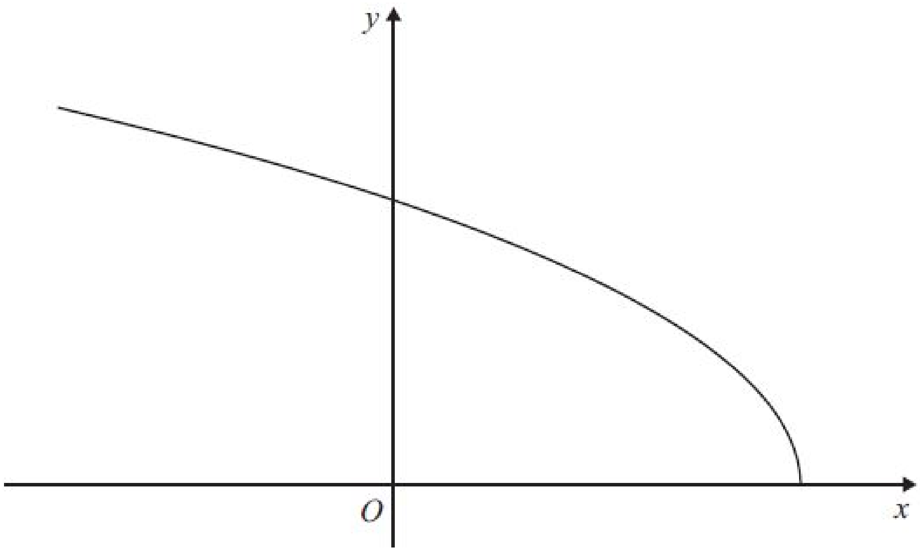


Figure 2

Figure 2 shows a sketch of the curve with parametric equations

$$x = 2\cos 2t, \qquad y = 6\sin t, \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

(a) Find the gradient of the curve at the point where $t = \frac{\pi}{3}$.

(4)

(b) Find a cartesian equation of the curve in the form

$$y = \mathrm{f}(x), \quad -k \leqslant x \leqslant k,$$

stating the value of the constant *k*.

(4)

(c) Write down the range of f(*x*).

(2)

(Total 10 marks)

Q15.

Relative to a fixed origin O, the point A has position vector $21\mathbf{i} - 17\mathbf{j} + 6\mathbf{k}$ and the point B has position vector $25\mathbf{i} - 14\mathbf{j} + 18\mathbf{k}$.

The line l has vector equation

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ c \\ -1 \end{pmatrix}$$

where a, b and c are constants and λ is a parameter.

Given that the point A lies on the line l,

(a) find the value of a.

Given also that the vector $\overrightarrow{AB}$ is perpendicular to l,

(3)

(b) find the values of b and c,

(5)

(c) find the distance AB.

(2)

The image of the point B after reflection in the line l is the point B'.

(d) Find the position vector of the point B'.

(2)

(Total 12 marks)

Q16.

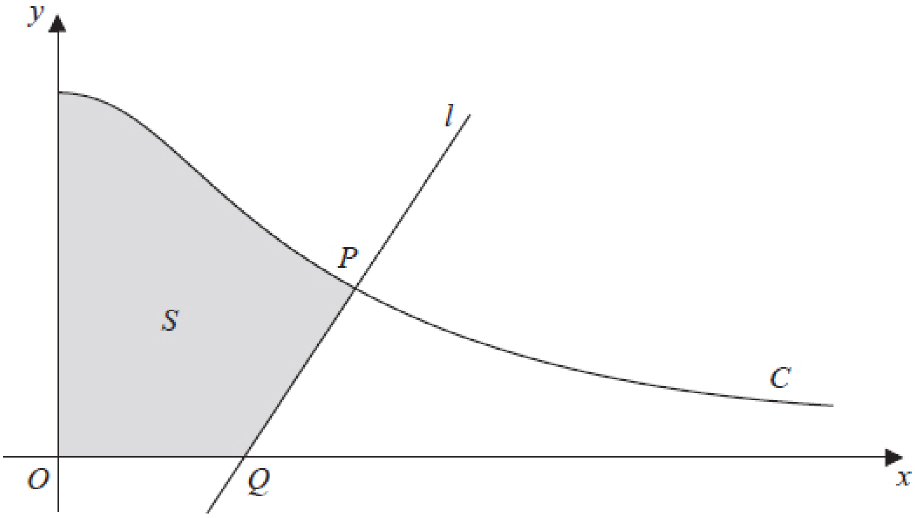


Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\tan\theta, \quad y = 4\cos^2\theta, \quad 0 \leq \theta < \tfrac{\pi}{2}$$

The point P lies on C and has coordinates (3, 2).

The line l is the normal to C at P. The normal cuts the x-axis at the point Q.

(a) Find the x coordinate of the point Q.

(6)

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the x-axis, the y-axis and the line l. This shaded region is rotated 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of the solid of revolution, giving your answer in the form $p\pi + q\pi^2$, where p and q are rational numbers to be determined.

[*You may use the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone*.]

(9)

(Total 15 marks)

Q17.

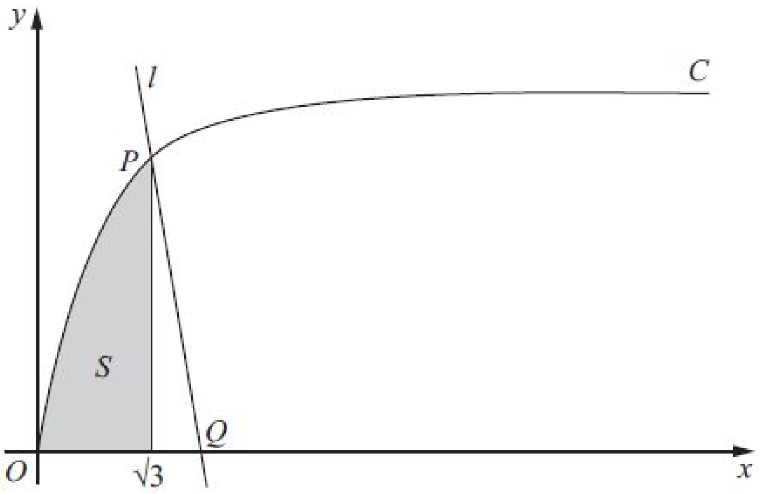


Figure 3

Figure 3 shows part of the curve *C* with parametric equations

$$x = \tan\theta, \qquad y = \sin\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point *P* lies on *C* and has coordinates $\left(\sqrt{3}, \frac{1}{2}\sqrt{3}\right)$.

(a) Find the value of θ at the point *P*.

(2)

The line *l* is a normal to *C* at *P*. The normal cuts the *x*-axis at the point *Q*.

(b) Show that *Q* has coordinates ($k\sqrt{3}$, 0), giving the value of the constant *k*.

(6)

The finite shaded region *S* shown in Figure 3 is bounded by the curve *C*, the line $x = \sqrt{3}$ and the *x*-axis. This shaded region is rotated through 2π radians about the *x*-axis to form a solid of revolution.

(c) Find the volume of the solid of revolution, giving your answer in the form $p\pi\sqrt{3} + q\pi^2$, where *p* and *q* are constants.

(7)

(Total 15 marks)

Q18.

The curve *C* has parametric equations

$$x = \ln t, \qquad y = t^2 - 2, \qquad t > 0$$

Find

(a) an equation of the normal to C at the point where $t = 3$,

(6)

(b) a cartesian equation of C.

(3)

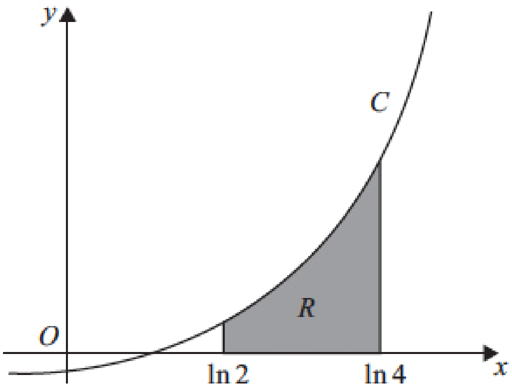


Figure 1

The finite area R, shown in Figure 1, is bounded by C, the x-axis, the line $x = \ln 2$ and the line $x = \ln 4$. The area R is rotated through 360° about the x-axis.

(c) Use calculus to find the exact volume of the solid generated.

(6)

(Total 15 marks)

Q19.

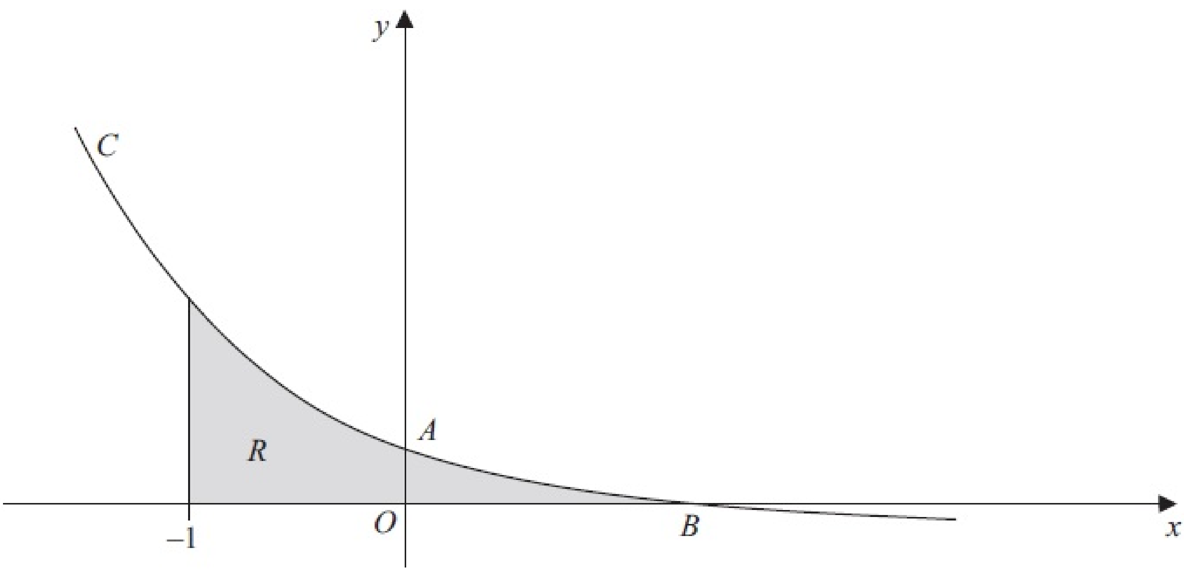


Figure 2

Figure 2 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) Show that A has coordinates (0, 3).

(2)

(b) Find the x coordinate of the point B.

(2)

(c) Find an equation of the normal to C at the point A.

(5)

The region R, as shown shaded in Figure 2, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R.

(6)

(Total 15 marks)

Q20.

Differentiate with respect to x, giving your answer in its simplest form,

(a) $x^2\ln(3x)$

(4)

(b) $\frac{\sin 4x}{x^3}$

(5)

(Total 9 marks)

Q21.

The point P is the point on the curve $x = 2\tan\left(y + \frac{\pi}{12}\right)$ with y-coordinate $\frac{\pi}{4}$.

Find an equation of the normal to the curve at P.

(7)

(Total 7 marks)

Q22.

(a) Find the value of $\frac{dy}{dx}$ at the point where x = 2 on the curve with equation

$$y = x^2 \sqrt{(5x - 1)}.$$

(6)

(b) Differentiate $\frac{\sin 2x}{x^2}$ with respect to x.

(4)

(Total 10 marks)

Q23.

Find the equation of the tangent to the curve $x = \cos(2y + \pi)$ at $\left(0, \frac{\pi}{4}\right)$.

Give your answer in the form $y = ax + b$, where a and b are constants to be found.

(6)

(Total 6 marks)

Q24.

The curve C has equation

$$y = \frac{3 + \sin 2x}{2 + \cos 2x}$$

(a) Show that

$$\frac{dy}{dx} = \frac{6 \sin 2x + 4 \cos 2x + 2}{(2 + \cos 2x)^2}$$

(4)

(b) Find an equation of the tangent to C at the point on C where $x = \frac{\pi}{2}$.

Write your answer in the form $y = ax + b$, where a and b are exact constants.

(4)

(Total 8 marks)

Q25.

A curve C has equation

$$y = \frac{3}{(5-3x)^2}, \quad x \neq \frac{5}{3}$$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(7)

(Total 7 marks)

Q26.

(i) Given that $y = \frac{\ln(x^2+1)}{x}$, find $\frac{dy}{dx}$.

(4)

(ii) Given that $x = \tan y$, show that $\frac{dy}{dx} = \frac{1}{1+x^2}$.

(5)

(Total 9 marks)

Q27.

(a) By writing $\sec x$ as $\frac{1}{\cos x}$, show that $\frac{d(\sec x)}{dx} = \sec x \tan x$.

(3)

Given that $y = e^{2x} \sec 3x$,

(b) find $\frac{dy}{dx}$.

(4)

The curve with equation $y = e^{2x} \sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b).

(c) Find the values of the constants a and b, giving your answers to 3 significant figures.

(4)

(Total 11 marks)

Q28.

(i) Differentiate with respect to x

(a) $x^2 \cos 3x$

(3)

(b) $\dfrac{\ln(x^2+1)}{x^2+1}$

(4)

(ii) A curve C has the equation

$$y = \sqrt{(4x+1)}, \quad x > -\tfrac{1}{4}, \; y > 0$$

The point P on the curve has x-coordinate 2. Find an equation of the tangent to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(6)

(Total 13 marks)

Q29.

The curve C has equation

$$y = (2x - 3)^5$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(a) the value of w,

(2)

(b) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants.

(5)

(Total 7 marks)

Q30.

(i) Differentiate with respect to x

(a) $y = x^3 \ln 2x$

(b) $y = (x + \sin 2x)^3$

(6)

Given that $x = \cot y$,

(ii) show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{1+x^2}$

(5)

(Total 11 marks)

Q31.

(a) Differentiate

$$\frac{\cos 2x}{\sqrt{x}}$$

with respect to x.

(3)

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

(3)

(c) Given $x = 2 \sin\left(\frac{y}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, simplifying your answer.

(4)

(Total 10 marks)

Q32.

The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x+1}{x-2}, \quad x > 2$$

(a) Show that

$$\mathrm{f}'(x) = \frac{-9}{(x-2)^2}$$

(3)

Given that P is a point on C such that $\mathrm{f}'(x) = -1$,

(b) find the coordinates of P.

(3)

(Total 6 marks)

Q33.

The curve C has equation $x = 8y \tan 2y$

The point P has coordinates $\left(\pi, \frac{\pi}{8}\right)$

(a) Verify that P lies on C.

(1)

(b) Find the equation of the tangent to C at P in the form $ay = x + b$, where the constants a and b are to be found in terms of π.

(7)

(Total 8 marks)

Q34.

A curve C has equation $y = e^{4x} + x^4 + 8x + 5$

(a) Show that the x coordinate of any turning point of C satisfies the equation

$$x^3 = -2 - e^{4x}$$

(3)

(b) On the axes given on page 5, sketch, on a single diagram, the curves with equations

(i) $y = x^3$,

(ii) $y = -2 - e^{4x}$

On your diagram give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

(4)

(c) Explain how your diagram illustrates that the equation $x^3 = -2 - e^{4x}$ has only one root.

(1)

The iteration formula

$$x_{n+1} = (-2 - e^{4x_n})^{\frac{1}{3}}, \quad x_0 = -1$$

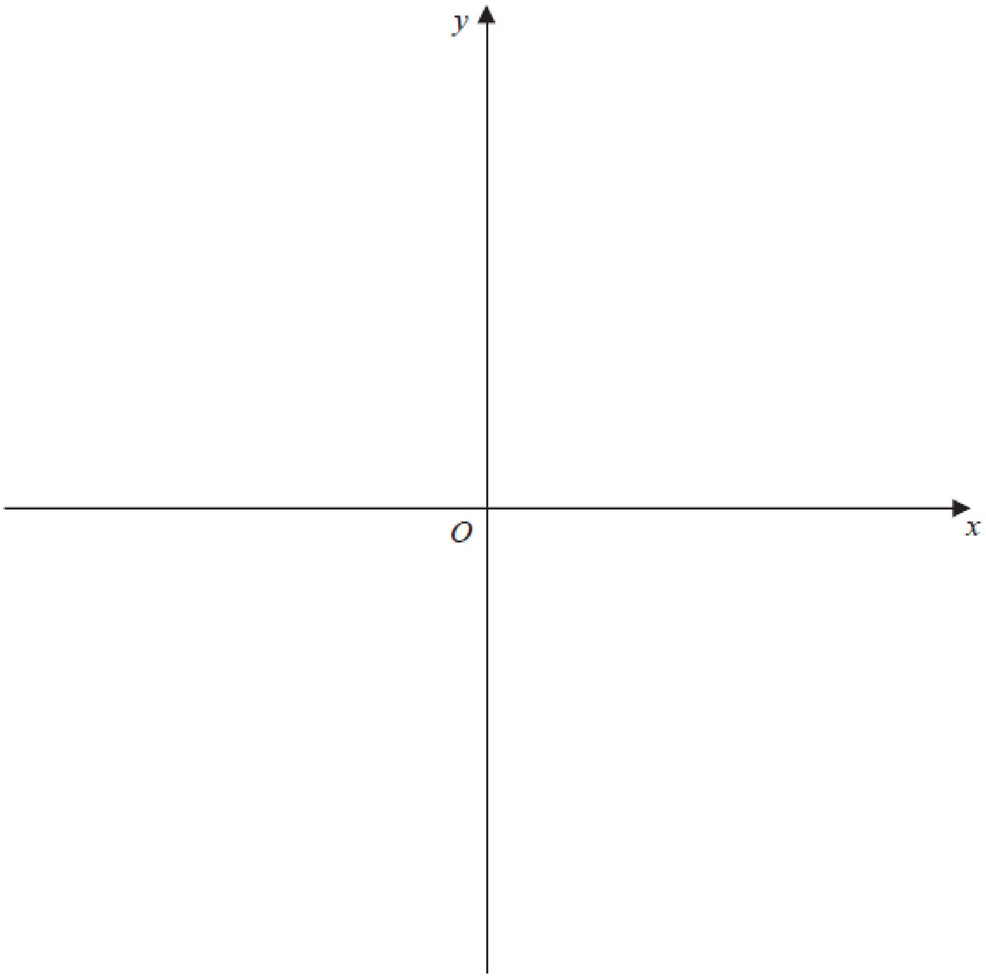
can be used to find an approximate value for this root.

(d) Calculate the values of x_1 and x_2, giving your answers to 5 decimal places.

(2)

(e) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve C.

(2)



(Total 12 marks)

Q35.

(i) (a) Show that $2 \tan x - \cot x = 5 \operatorname{cosec} x$ may be written in the form

$$a \cos^2 x + b \cos x + c = 0$$

stating the values of the constants a, b and c.

(4)

(b) Hence solve, for $0 \leq x < 2\pi$, the equation

$$2 \tan x - \cot x = 5 \operatorname{cosec} x$$

giving your answers to 3 significant figures.

(4)

(ii) Show that

$$\tan \theta + \cot \theta \equiv \lambda \operatorname{cosec} 2\theta, \quad \theta \pm \frac{n\pi}{2}, \; n \in \mathbb{Z}$$

stating the value of the constant λ.

(4)

(Total 12 marks)

Q36.

The curve C has the equation $2x + 3y^2 + 3x^2 y = 4x^2$.

The point P on the curve has coordinates $(-1, 1)$.

(a) Find the gradient of the curve at P.

(5)

(b) Hence find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c
are integers.

(3)

(Total 8 marks)

Q37.

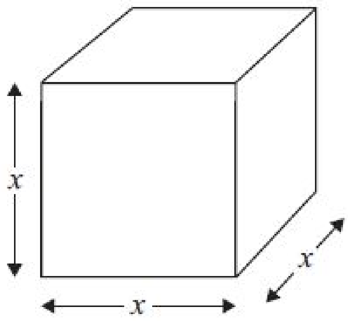


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated.
At time t seconds, the length of each edge of the cube is x cm,
and the volume of the cube is V cm^3.

(a) Show that $\dfrac{dV}{dx} = 3x^2$

(1)

Given that the volume, V cm^3, increases at a constant rate of 0.048 cm^{3}s^{-1},

(b) find $\dfrac{dx}{dt}$, when $x = 8$

(2)

(c) find the rate of increase of the total surface area of the cube, in cm^2s^{-1}, when $x = 8$

(3)

(Total 6 marks)

Q38.

The curve C has equation

$$16y^3 + 9x^2y - 54x = 0$$

(a) Find $\dfrac{dy}{dx}$ in terms of x and y.

(5)

(b) Find the coordinates of the points on C where $\dfrac{dy}{dx} = 0$.

(7)

(Total 12 marks)

Q39.

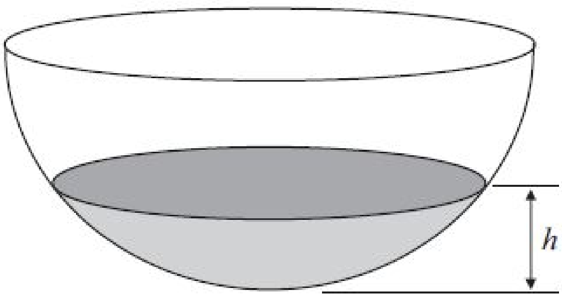


Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl. When the depth of the water is h m, the volume V m^3 is given by

$$V = \frac{1}{12}\pi h^2(3-4h), \qquad 0 \leqslant h \leqslant 0.25$$

(a) Find, in terms of

π, $\frac{dV}{dh}$ when h = 0.1

(4)

Water flows into the bowl at a rate of $\frac{\pi}{800}$ $m^3 s^{-1}$.

(b) Find the rate of change of h, in m s^{-1}, when h = 0.1

(2)

(Total 6 marks)

Q40.

Find the gradient of the curve with equation

$$\ln y = 2x \ln x, \quad x > 0, \ y > 0$$

at the point on the curve where x = 2. Give your answer as an exact value.

(7)

(Total 7 marks)

Q41.

A curve C has the equation $y^2 - 3y = x^3 + 8$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(4)

(b) Hence find the gradient of C at the point where y = 3.

(3)

(Total 7 marks)

Q42.

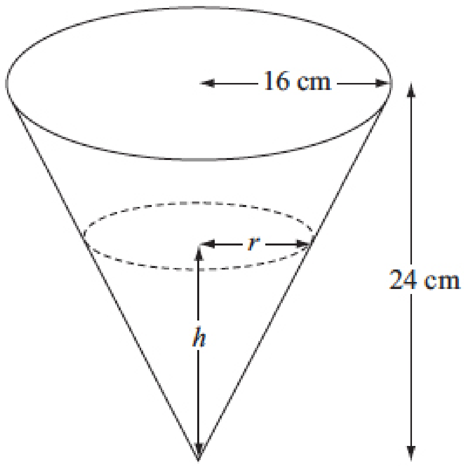


Figure 2

A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in Figure 2. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(a) Show that $V = \frac{4\pi h^3}{27}$.

(2)

[The volume V of a right circular cone with vertical height h and base radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.*]*

Water flows into the container at a rate of 8 cm^3 s^{-1}.

(b) Find, in terms of π, the rate of change of h when $h = 12$.

(5)

(Total 7 marks)

Q43.

The current, I amps, in an electric circuit at time t seconds is given by

$$I = 16 - 16(0.5)^t, \quad t \geqslant 0$$

Use differentiation to find the value of $\frac{\mathrm{d}I}{\mathrm{d}t}$ when $t = 3$.

Give your answer in the form ln a, where a is a constant.

(5)

(Total 5 marks)

Q44.

A curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (3, 2).

(7)

(Total 7 marks)

Q45.

The curve C has the equation

$$\cos 2x + \cos 3y = 1, \qquad -\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}, \quad 0 \leqslant y \leqslant \frac{\pi}{6}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(3)

The point P lies on C where $x = \frac{\pi}{6}$.

(b) Find the value of y at P.

(3)

(c) Find the equation of the tangent to C at P, giving your answer in the form $ax + by + c\pi = 0$, where a, b and c are integers.

(3)

(Total 9 marks)

Q46.

The area A of a circle is increasing at a constant rate of 1.5 $cm^2\ s^{-1}$. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is 2 cm^2.

(5)

(Total 5 marks)

Q47.

The curve C has the equation $ye^{-2x} = 2x + y^2$.

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

The point P on C has coordinates (0, 1).

(b) Find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(4)

(Total 9 marks)

Q48.

A curve is described by the equation

$$x^2 + 4xy + y^2 + 27 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

A point Q lies on the curve.

The tangent to the curve at Q is parallel to the y-axis.

Given that the x coordinate of Q is negative,

(b) use your answer to part (a) to find the coordinates of Q.

(7)

(Total 12 marks)

Q49.

The curve C has equation

$$3^{x-1} + xy - y^2 + 5 = 0$$

Show that $\frac{dy}{dx}$ at the point (1, 3) on the curve C can be written in the form $\frac{1}{\lambda}\ln(\mu e^3)$, where λ and μ are integers to be found.

(7)

(Total 7 marks)

Q50.

A curve C has the equation

$$x^3 + 2xy - x - y^3 - 20 = 0$$

(a) Find $^{dy}/_{dx}$ in terms of x and y.

(5)

(b) Find an equation of the tangent to C at the point (3, −2), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(2)

(Total 7 marks)

Q51.

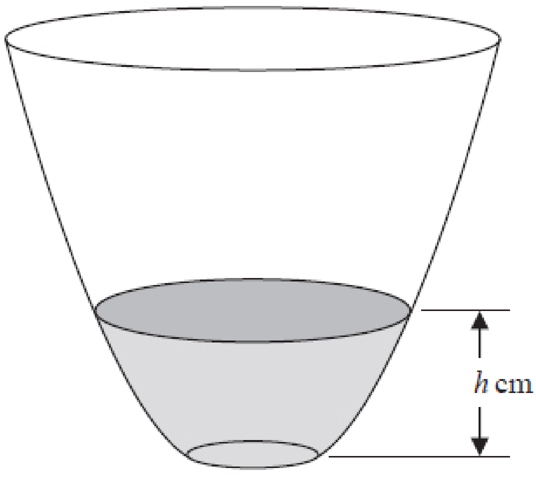


Figure 2

A vase with a circular cross-section is shown in Figure 2. Water is flowing into the vase.

When the depth of the water is h cm, the volume of water V cm^3 is given by

$$V = 4\pi h(h + 4), \quad 0 \le h \le 25$$

Water flows into the vase at a constant rate of 80π cm^{3}s^{-1}

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 6$

(5)

(Total 5 marks)

Q52.

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

(a) Find ${}^{dy}/_{dx}$ in terms of x and y, fully simplifying your answer.

(5)

(b) Find the values of y for which ${}^{dy}/_{dx} = 0$

(5)

(Total 10 marks)

Q53.

At time t seconds the radius of a sphere is r cm, its volume is V cm^3 and its surface area is S cm^2.

[*You are given that* $V = {}^4/_3\, \pi r^3$ *and that* $S = 4\pi r^2$]

The volume of the sphere is increasing uniformly at a constant rate of 3 cm^3 s^{-1}.

(a) Find ${}^{dr}/_{dt}$ when the radius of the sphere is 4 cm, giving your answer to 3 significant figures.

(4)

(b) Find the rate at which the surface area of the sphere is increasing when the radius is 4 cm.

(2)

(Total 6 marks)

Q54.

(a) Given that $y = 3^x$, find $\frac{dy}{dx}$

(1)

(b) Find an equation of the tangent to the curve

$$y = 3^x - 3^{-x} + 2$$

at the point (0, 2).

(5)

(Total 6 marks)

Q55.

A circular cylinder has a perpendicular height equal to the radius of the base r cm.

Given that the volume of this cylinder is V cm^3 ,

(a) show that $V = \pi r^3$

(1)

Given that r varies with time,

(b) find $\frac{dV}{dr}$ in terms of r.

(1)

The rate of change of the volume of the cylinder at time t seconds, $\frac{dV}{dt}$ cm^{3}s^{-1}, is given by

$$\frac{dV}{dt} = \frac{2t}{2+t^2}$$

Given that $V = 3$ when $t = 0$,

(c) find V in terms of t.

(4)

(d) When $t = 1$, find r. Give your answer to 3 significant figures.

(4)

(e) Using the Chain Rule, or otherwise, find $\frac{dr}{dt}$ in terms of r and t.

(2)

(f) When $t = 1$, find the value of $\frac{dr}{dt}$. Give your answer to 3 significant figures.

(2)

(Total 14 marks)

Q56.

The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = pe^{-kt}$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(a) Write down the value of p.

(1)

(b) Show that $k = \frac{1}{4}\ln 3$.

(4)

(c) Find the value of t when $\frac{dm}{dt} = -0.6\ln 3$.

(6)

(Total 11 marks)

Q57.

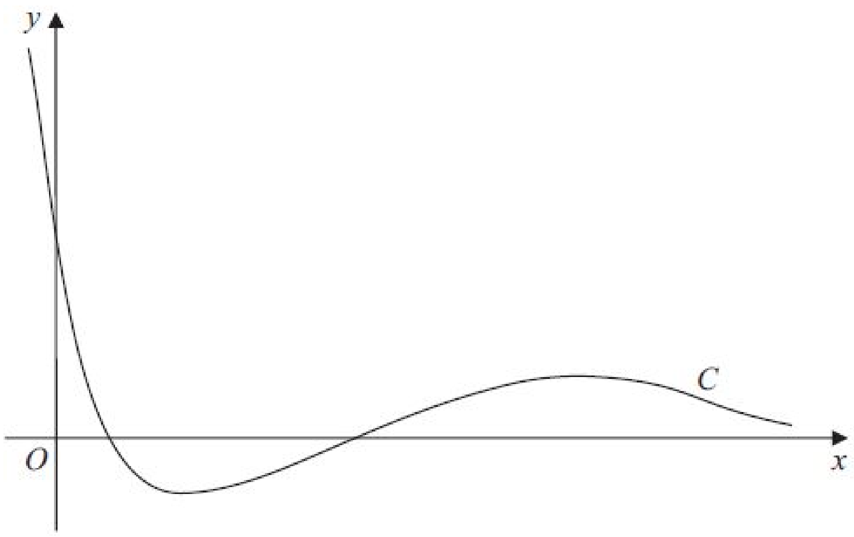


Figure 1

Figure 1 shows a sketch of the curve C with the equation $y = (2x^2 - 5x + 2)e^{-x}$.

(a) Find the coordinates of the point where C crosses the y-axis.

(1)

(b) Show that C crosses the x-axis at $x = 2$ and find the x-coordinate of the other point where C crosses the x-axis.

(3)

(c) Find $\frac{dy}{dx}$.

(3)

(d) Hence find the exact coordinates of the turning points of C.

(5)

(Total 12 marks)

Q58.

A rare species of primrose is being studied. The population, P, of primroses at time t years after the study started is modelled by the equation

$$P = \frac{800e^{0.1t}}{1 + 3e^{0.1t}}, \quad t \geq 0, t \in \mathbb{R}.$$

(a) Calculate the number of primroses at the start of the study.

(2)

(b) Find the exact value of t when $P = 250$, giving your answer in the form $a \ln(b)$ where a and b are integers.

(4)

(c) Find the exact value of $\frac{dP}{dt}$ when $t = 10$. Give your answer in its simplest form.

(4)

(d) Explain why the population of primroses can never be 270

(1)

(Total 11 marks)

Q59.

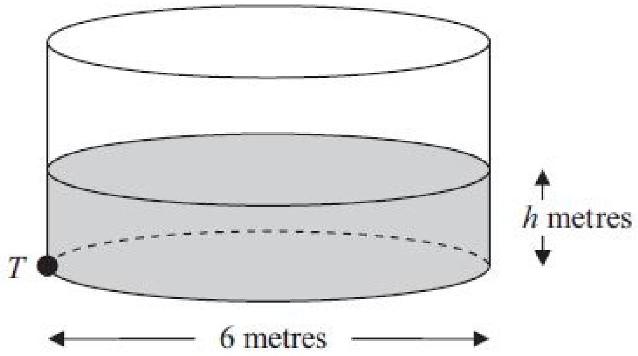


Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that t minutes after the tap has been opened

$$75\frac{\mathrm{d}h}{\mathrm{d}t} = (4 - 5h)$$

(5)

When $t = 0$, $h = 0.2$

(b) Find the value of t when $h = 0.5$

(6)

(Total 11 marks)

Q60.

$$\mathrm{f}(x) = 25x^2\mathrm{e}^{2x} - 16, \qquad x \in \mathbb{R}$$

(a) Using calculus, find the exact coordinates of the turning points on the curve with equationy = f(x).

(5)

(b) Show that the equation f(x) = 0 can be written as $x = \pm\frac{4}{5}\mathrm{e}^{-x}$

(1)

The equation f(x) = 0 has a root α, where $\alpha = 0.5$ to 1 decimal place.

(c) Starting with $x_0 = 0.5$, use the iteration formula

$$x_{n+1} = \frac{4}{5}\mathrm{e}^{-x_n}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

(d) Give an accurate estimate for α to 2 decimal places, and justify your answer.

(2)

(Total 11 marks)

Q61.

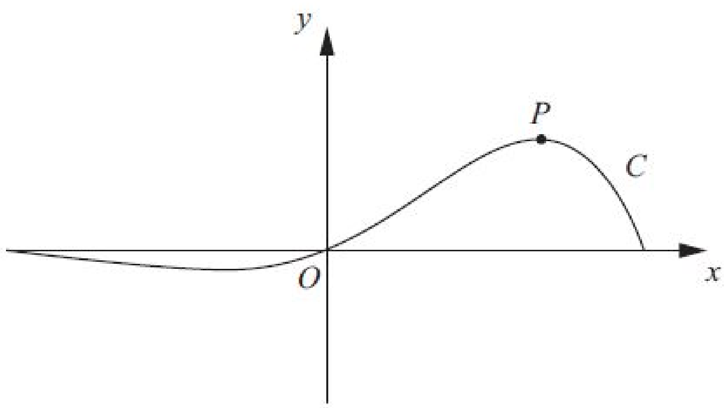


Figure 1

Figure 1 shows a sketch of the curve *C* which has equation

$$y = e^{x\sqrt{3}} \sin 3x, \quad -\frac{\pi}{3} \leqslant x \leqslant \frac{\pi}{3}$$

(a) Find the x coordinate of the turning point *P* on *C*, for which $x > 0$
Give your answer as a multiple of π.

(6)

(b) Find an equation of the normal to *C* at the point where $x = 0$

(3)

(Total 9 marks)

Q62.

(a) Differentiate with respect to x,

(i) $x^{\frac{1}{2}} \ln(3x)$

(ii) $\dfrac{1-10x}{(2x-1)^5}$, giving your answer in its simplest form.

(6)

(b) Given that $x = 3\tan 2y$ find ${}^{dy}/_{dx}$ in terms of x.

(5)

(Total 11 marks)

Q63.

Given that

$$x = \sec^2 3y, \qquad 0 < y < \frac{\pi}{6}$$

(a) find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6x(x-1)^{\frac{1}{2}}}$$

(4)

(c) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x. Give your answer in its simplest form.

(4)

(Total 10 marks)

Q64.

Joan brings a cup of hot tea into a room and places the cup on a table. At time t minutes after Joan places the cup on the table, the temperature, θ°C, of the tea is modelled by the equation

$$\theta = 20 + Ae^{-kt},$$

where A and k are positive constants.

Given that the initial temperature of the tea was 90°C,

(a) find the value of A.

(2)

The tea takes 5 minutes to decrease in temperature from 90°C to 55°C.

(b) Show that $k = \frac{1}{5}\ln 2$.

(3)

(c) Find the rate at which the temperature of the tea is decreasing at the instant when $t = 10$. Give your answer, in °C per
minute, to 3 decimal places.

(3)

(Total 8 marks)

Q65.

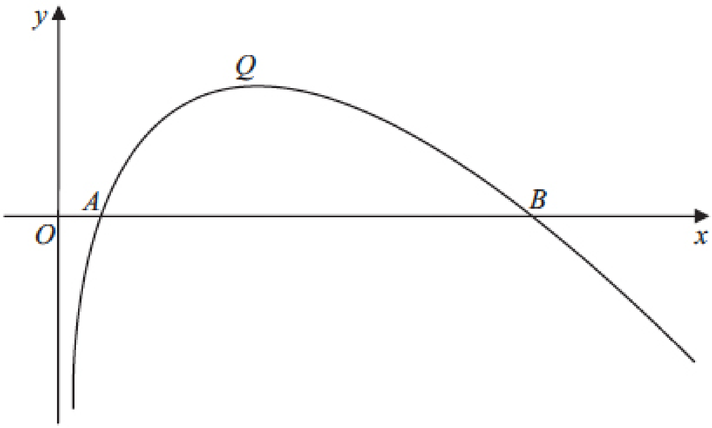


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \text{f}(x)$, where

$$\text{f}(x) = (8 - x)\ln x, \qquad x > 0$$

The curve cuts the x-axis at the points A and B and has a maximum turning point at Q, as shown in Figure 1.

(a) Write down the coordinates of A and the coordinates of B.

(2)

(b) Find $\text{f}'(x)$.

(3)

(c) Show that the x-coordinate of Q lies between 3.5 and 3.6

(2)

(d) Show that the x-coordinate of Q is the solution of

$$x = \frac{8}{1 + \ln x}$$

(3)

To find an approximation for the x-coordinate of Q, the iteration formula

$$x_{n+1} = \frac{8}{1+\ln x_n}$$

is used.

(e) Taking $x_0 = 3.55$, find the values of x_1, x_2 and x_3.
Give your answers to 3 decimal places.

(3)

(Total 13 marks)

Q66.

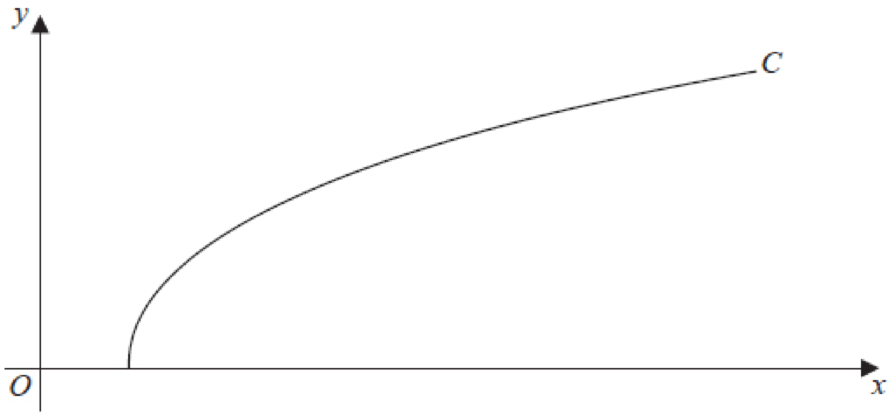


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 27\sec^3 t, \quad y = 3\tan t, \qquad 0 \leqslant t \leqslant \frac{\pi}{3}$$

(a) Find the gradient of the curve C at the point where $t = \frac{\pi}{6}$

(4)

(b) Show that the cartesian equation of C may be written in the form

$$y = (x^{\frac{2}{3}} - 9)^{\frac{1}{2}}, \quad a \leq x \leq b$$

stating the values of a and b.

(3)

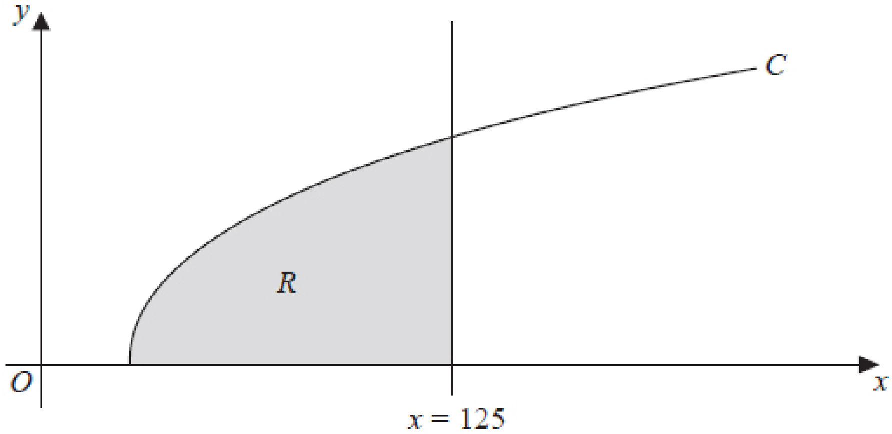


Figure 3

The finite region R which is bounded by the curve C, the x-axis and the line $x = 125$ is shown shaded in Figure 3. This region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Use calculus to find the exact value of the volume of the solid of revolution.

(5)

(Total 12 marks)

Q67.

The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt - 1)(5000 - N)}{t} \qquad t > 0, \quad 0 < N < 5000$$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(a) By solving the differential equation, show that

$$N = 5000 - Ate^{-kt}$$

where A is a positive constant.

(5)

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(b) Find the exact value of the constant A and the exact value of the constant k.

(4)

(c) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish.

(1)

(Total 10 marks)

Q68.

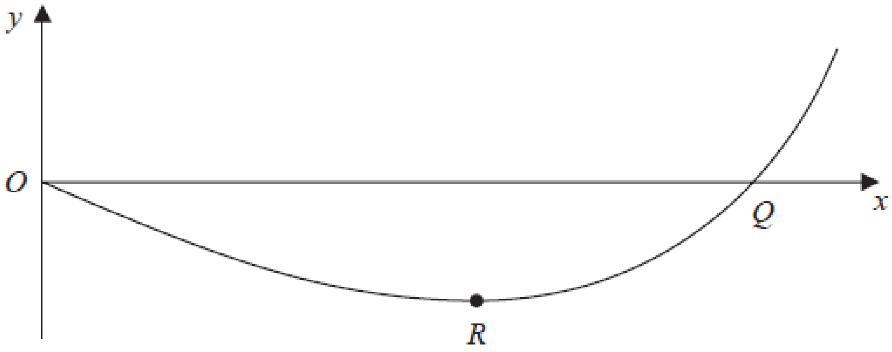


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = 2\cos\left(\frac{1}{2}x^2\right) + x^3 - 3x - 2$$

The curve crosses the x-axis at the point Q and has a minimum turning point at R.

(a) Show that the x coordinate of Q lies between 2.1 and 2.2

(2)

(b) Show that the x coordinate of R is a solution of the equation

$$x = \sqrt{1 + \frac{2}{3}x\sin\left(\frac{1}{2}x^2\right)}$$

(4)

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \frac{2}{3}x_n\sin\left(\frac{1}{2}{x_n}^2\right)}, \quad x_0 = 1.3$$

(c) find the values of x_1 and x_2 to 3 decimal places.

(2)

(Total 8 marks)

Q69.

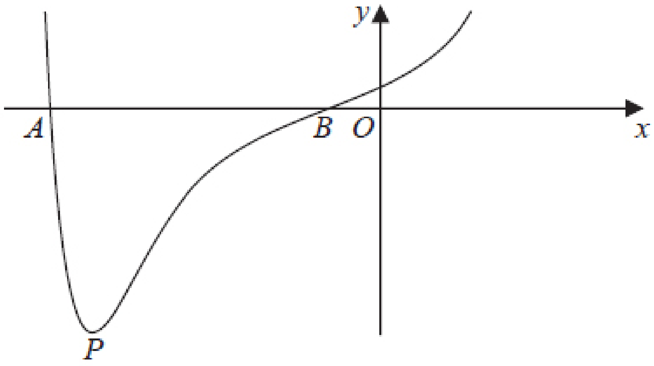


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}$$

The curve cuts the x-axis at points A and B as shown in Figure 2.

(a) Calculate the x coordinate of A and the x coordinate of B, giving your answers to 3 decimal places.

(2)

(b) Find $\mathrm{f}'(x)$.

(3)

The curve has a minimum turning point at the point P as shown in Figure 2.

(c) Show that the x coordinate of P is the solution of

$$x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)}$$

(3)

(d) Use the iteration formula

$$x_{n+1} = -\frac{3(2x_n^2 + 1)}{2(x_n^2 + 2)}, \qquad \text{with } x_0 = -2.4,$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

The x coordinate of P is α.

(e) By choosing a suitable interval, prove that $\alpha = -2.43$ to 2 decimal places.

(2)

(Total 13 marks)

Q70.

(a) Express 2cos 3*x* − 3sin 3*x* in the form *R* cos (3*x* + α), where *R* and α are constants, *R* > 0 and $0 < \alpha < \frac{\pi}{2}$. Give your answers to 3 significant figures.

(4)

$$f(x) = e^{2x} \cos 3x$$

(b) Show that f '(*x*) can be written in the form

$$f'(x) = R e^{2x} \cos(3x + \alpha)$$

where *R* and α are the constants found in part (a).

(5)

(c) Hence, or otherwise, find the smallest positive value of *x* for which the curve with equation *y* = f(*x*) has a turning point.

(3)

(Total 12 marks)

Q71.

(a) Given that

$$\frac{d}{dx}(\cos x) = -\sin x$$

show that $\frac{d}{dx}$ (sec *x*) sec *x* tan *x*.

(3)

Given that

$$x = \sec 2y$$

(b) find $\frac{dx}{dy}$ in terms of *y*.

(2)

(c) Hence find $\frac{dy}{dx}$ in terms of *x*.

(4)

(Total 9 marks)